Example. Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps

(1) Identify the domain of f and symmetries the curve may have.
$(-\infty, \infty)$ since it is a polynomial. There are no symmetries (neither even nor odd). Oddness fails because $f(0) \neq 0$. Evenness fails because

$$f(-x) = (-x)^4 - 4(-x)^3 + 10 = x^4 + 4x^3 + 10 \neq f(x).$$

(2) Intercepts.
y-intercept: $f(0) = 10$, and x intercept is hard to find here. But don't worry.

(3) Find the derivatives f' and f''.

$$f'(x) = 4x^3 - 12x^2,$$

and

$$f''(x) = 12x^2 - 24x.$$

(4) Find the critical points of f, if any, and identify the function's behaviour at each one.
Critical points satisfy

$$0 = f'(x) = 4x^3 - 12x^2 \implies 4x^2(x-3) = 0 \implies x = 0, \quad x = 3.$$

So, our domain is chopped down into $(-\infty, 0), (0, 3)$ and $(3, \infty)$. Note that $f'(x) = 4x^2(x-3)$ which means the sign of f' is completely determined by the sign of $x - 3$ (as $4x^2 > 0$ for all x on any of these three intervals).

(5) Find where the curve is increasing and where it is decreasing.
- On $(-\infty, 0)$, $x - 3 < 0$. Therefore, f is decreasing on $(-\infty, 0)$.
- On $(0, 3)$, $x - 3 < 0$. Therefore, f is decreasing on $(0, 3)$.
- On $(3, \infty)$, $x - 3 > 0$. Therefore, f is increasing on $(3, \infty)$.

From this information, we know that

	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
f'	$-$	$-$	$+$
f	dec	dec	inc

which means $x = 3$ is a **local minimum**. We can't say anything about $x = 0$ as of now.

(6) Find the points of inflection, if any occur, and determine the concavity of the curve.
The candidates for points of inflection satisfy

$$0 = f''(x) = 12x^2 - 24x = 12x(x-2) \implies x = 0, \quad x = 2.$$

Now, let's check concavity changes. The intervals we check now is $(-\infty, 0)$, $(0, 2)$ and $(2, \infty)$. Note that the sign of $f''(x)$ depends on both terms $12x$ and $(x - 2)$.
- On $(-\infty, 0)$, $12x < 0$ and $(x - 2) < 0$. Thus $f''(x) = 12x(x-2) > 0$, which means the graph of f is concaving up.
- On $(0, 2)$, $12x > 0$ and $(x - 2) < 0$. Thus $f''(x) = 12x(x-2) < 0$, which means the graph of f is concaving down.
- On $(2, \infty)$, $12x > 0$ and $x - 2 > 0$. Thus $f''(x) = 12x(x-2) > 0$, which means the graph of f is concaving up.
  From this information, we know that

	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
f''	$+$	$-$	$+$
f	concave up	concave down	concave up

This means that both $x = 0$ and $x = 2$ are points of inflection since there is concavity change respectively.

(7) Identify any asymptotes that may exist.

No asymptotes. However, it is useful to check behaviour at infinity.

$$\lim_{x\to\infty} f(x) = \infty, \quad \lim_{x\to-\infty} f(x) = \infty,$$

which means the graph of the function at both infinities are blowing up upwards. Noting that the concavity at these two extreme ends are also concaving up, the blowing up is like a bowl.

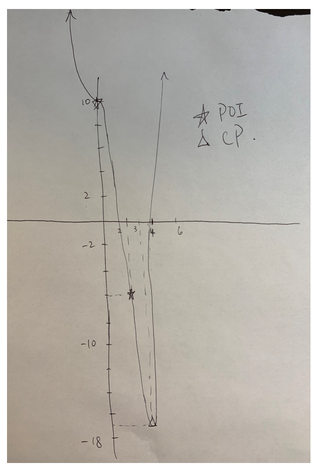
(8) Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

So, the key points at $x = 0$, 2 and 3. At least, we need to label their locations,

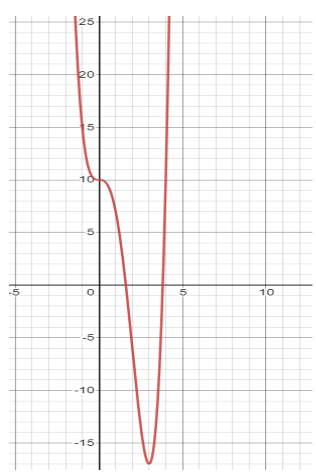
$$f(0) = 10, \quad f(2) = -6, \quad f(3) = -17.$$

You may find the graph of f on $(3,\infty)$ hard to plot, since it only gives increasing and concave up. However, you should be able to determine possibly where the root is, because the function has to blow up $+\infty$ which at $f(3) = -17 < 0$. I tested, $f(4) = 10$, which means there is a point $c \in (3,4)$ such that $f(c) = 0$.

You may also find the graph of f on $(-\infty, 0)$ hard to plot, since it is only decreasing and concave up. But note that, on the left of $x = 0$, x^4 dominates, so it must blow up like x^4, which is very fast – it sky rockets.



My attempt



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You probably can't tell if $x = 2$ has a concavity change, since the function has quite a big derivative, and it's hard to draw. But you can definitely see the concavity change at $x = 0$. However, showing the work that $x = 2$ is an inflection point should give you lots of credits. The rest is all artistic.